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Extension of Kramers' symbolic method to SU(4)

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Abstract. The invariant symbolic method initiated by Kramers in SU(2) is extended here to the now physically interesting group SU(4) by a projection technique which is described and applied to a reduction of direct products giving closed formulae for the multiplicities of some classes of representations.

1. Introduction

Many years ago, Kramers (1930, 1931) initiated a method of calculation called the 'symbolic', or 'spinor invariants', method in atomic physics. With this tool he was able to obtain very easily numerous results pertaining to the rotational symmetry of physical systems. Selection rules, intensity rules for atomic spectra and Clebsch–Gordan coefficients were obtained in this way (Brinkman 1956, Heine 1960).

I was able (Jasselette 1967a, b, c) to extend this method to the, at that time, more fashionable group SU(3). The difficulties circumvented were

(i) the occurence of two non-equivalent fundamental representations 3 and $\bar{3}$ and

(ii) the trace condition for irreducibility.

The first one gives only extra work in handling the symbols, but the second one necessitates a projection technique from some reducible space (let us call it Kramers' space) to the irreducible representation required.

With the advent of charm (Gaillard *et al* 1975), SU(4) is becoming important for physics and it seems interesting to apply Kramers' method to this larger group. Apart from the two difficulties already met in SU(3) a third one complicates the game.

(iii) SU(4) is a rank-three group. It has irreducible representations of mixed symmetry and three fundamental representations 4, $\bar{4}$ and 6.

Here I will extend the projection technique already mentioned to meet this new difficulty, propose a choice for Kramers' spaces adapted for the different kinds of irreducible representations, define projectors going from the former to the latter, and give some results obtained in the reduction of direct products with this technique.

2. Quick review of the symbolic method

(1) The method is based on the transformation properties of physical quantities under the action of a given group. These quantities are represented by *symbolic tensors* having the same behaviour and required to be of monomial form:

$$u^{n_1}d^{n_2}\dots\bar{u}^{m_1}\bar{d}^{m_2}\dots \tag{1}$$

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(u, d, ... are components of a vector of the fundamental $\boldsymbol{N}$ representation; $\bar{u}$, $\bar{d}$, ... are components of a contragradient vector).

(2) With these symbols and one or several auxiliary vector(s) an *invariant* is built.

(3) The expected relations between physical quantities are similarly expressed as relations between the symbols.

(4) The invariant is reduced to a polynomial of basic invariants by virtue of two main theorems of invariant theory (Weyl 1946). The basic invariants in SU(N) are

$$(\bar{X}, Y) = \sum_{i}^{N} \bar{X}_i Y_i \tag{2}$$

and the determinants

$$|X, Y, \ldots, Z| = \sum_{P} \operatorname{sign} P X_{i_1} Y_{i_2} \ldots Z_{i_N} \tag{3}$$

$$|\bar{X}, \bar{Y}, \ldots, \bar{Z}| = \sum_{P} \operatorname{sign} P \bar{X}_{i_1} \bar{Y}_{i_2} \ldots \bar{Z}_{i_N}. \tag{4}$$

The theorems state that every invariant polynomial is a polynomial of basic invariants and that one relation exists between the latter, namely

$$\begin{vmatrix} (\bar{X}, X) & \ldots & (\bar{X}, Z) \\ (\bar{Y}, X) & \ldots & (\bar{Y}, Z) \\ \vdots & & \\ (\bar{Z}, X) & \ldots & (\bar{Z}, Z) \end{vmatrix} = |X, \ldots, Z|\,|\bar{X}, \ldots, \bar{Z}|. \tag{5}$$

(5) The coefficients of the different powers of the auxiliary vector's (s') components are compared in the forms taken by the invariant respectively before and after application of the theorems. Their identification leads to the required relations provided normalisation and phases are duly considered.

(6) The physical signification of the relations deduced is established.

3. The lazy calculator's progress in SU(4)

For physics the interesting representations are the irreducible ones. But in general in SU(4) these representations have no monomial basis so that the original method is not applicable. I suggest the following remedy for this difficulty.

(1) Leave the irreducible representation and go into a suitably chosen reducible monomial representation in a bigger space (Kramers' space);

(2) make the calculations in this bigger and easier space;

(3) define a projection operator from Kramers' space to the irreducible representation of interest in order to collect the desired results. This can also be viewed as defining a different metric in Kramers' space so that the representation it supports is irreducible.

4. Choice of Kramers' space and definition of projectors

4.1. The case of the irreducible representation $(n, 0, m)$†

This representation can be obtained in the reduction of the direct product of n basic representations 4 and m basic representations $\bar{4}$. A good candidate for Kramers' space in this case is the doubly symmetric representation $\{n, 0, m\}$ defined by the basis

$$\begin{matrix} n_i \\ 0 \\ m_i \end{matrix} = u^{n_1} d^{n_2} s^{n_3} c^{n_4} \bar{u}^{m_1} \bar{d}^{m_2} \bar{s}^{m_3} \bar{c}^{m_4} \tag{6}$$

$$\sum_i^4 n_i = n \qquad \sum_i^4 m_i = m.$$

Here also u, d, s, c stand for the four components of some vector in the fundamental **4** representation and $\bar{u}, \bar{d}, \bar{s}, \bar{c}$ for the components of a vector in $\bar{4}$.

Application of the two theorems quoted above and some combinatories (expansion of powers of polynomials) similar to those encountered in SU(3) (see Jasselette (1967a, b, c) for details) give the matrix elements of the projector from $\{n, 0, m\}$ to $(n, 0, m)$:

$$\begin{matrix} n'_i \\ 0 \\ m'_i \end{matrix} \; P \; \begin{matrix} n_i \\ 0 \\ m_i \end{matrix} = \frac{\Pi_{i=1}^4 n_i!\, m_i!\, n'_i!\, m'_i!}{n!\, m!} \sum_{k=0}^{m} C_k \sum_{\kappa_i \lambda_i \mu_i \nu_i} \frac{(n-k)!\,(m-k)!\,(k!)^2}{\Pi_{i=1}^4 \kappa_i!\, \lambda_i!\, \mu_i!\, \nu_i!} \tag{7}$$

with

$$\sum \kappa_i = n-k \qquad \sum \lambda_i = m-k \qquad \sum \mu_i = \sum \nu_i = k.$$

The constants C_k are $(m+1)$ in number, as can be deduced from the fact that $\{n, 0, m\}$ contains $(m+1)$ irreducible representations. They can be obtained by cancelling the traces

$$P\left[\begin{vmatrix} R & n-R & 0 & 0 \\ R & 0 & m-R & 0 \end{vmatrix} + \begin{vmatrix} R-1 & n-R+1 & 0 & 0 \\ R-1 & 1 & m-R & 0 \end{vmatrix} \right.$$
$$\left. + \begin{vmatrix} R-1 & n-R & 1 & 0 \\ R-1 & 0 & m-R+1 & 0 \end{vmatrix} + \begin{vmatrix} R-1 & n-R & 0 & 1 \\ R-1 & 0 & m-R & 1 \end{vmatrix}\right] \tag{8}$$

with $R = 1, \ldots, m$. Lengthy but easy calculations give

$$C_k = (-1)^k \binom{n}{k} \binom{m}{k} \binom{n+m+2}{k}^{-1} C_0$$
$$= \frac{(-1)^k n!\, m!\, (n+m+2-k)!}{(n-k)!\,(m-k)!\, k!\,(n+m+2)!} C_0. \tag{9}$$

C_0 can be chosen to be equal to 1.

† The standard notation $(\lambda_1, \lambda_2, \lambda_3)$ for an irreducible representation gives the differences in length of the rows in the Young tableau. It also gives the way the highest weight of the representation can be obtained from those of the fundamental representations 4, 6 and $\bar{4}$.

4.2. The case of the irreducible representation $(0, \lambda, 0)$

For the irreducible representation $(0, \lambda, 0)$, which is typical for SU(4), I suggest to choose as Kramers' space the product

$$(\lambda, 0, 0) \otimes (\lambda, 0, 0)$$

with a base

$$\left| \begin{matrix} n_i \\ N_i \\ 0 \end{matrix} \right\rangle = u^{n_1} d^{n_2} s^{n_3} c^{n_4} U^{N_1} D^{N_2} S^{N_3} C^{N_4} \qquad \sum n_i = \sum N_i = \lambda. \tag{10}$$

This reducible representation decomposes in

$$\begin{aligned} (\lambda, 0, 0) \otimes (\lambda, 0, 0) &= \bigoplus_{k=0}^{\lambda} (2\lambda - 2k, k, 0) \\ &= (2\lambda, 0, 0) \oplus (2\lambda - 2, 1, 0) \oplus \ldots \oplus (0, \lambda, 0). \end{aligned} \tag{11}$$

Different projectors are defined by

$$\begin{aligned} \begin{matrix} n_i' \\ N_i' \\ 0 \end{matrix} \; P \; \begin{matrix} n_i \\ N_i \\ 0 \end{matrix} &= \frac{\Pi_i n_i! N_i! n_i'! N_i'!}{[(2\lambda)!]^2} \sum_\gamma D^\gamma_{kk'} \gamma! (k + k' + \gamma - 2\lambda)! (2\lambda - k - \gamma)! \\ &\quad \times (2\lambda - k' - \gamma)! \sum_{\alpha_i \beta_i \gamma_i \delta_i} \frac{1}{\Pi_i \alpha_i! \beta_i! \gamma_i! \delta_i!} \end{aligned} \tag{12}$$

where k and k' are fixed by

$$k = \sum N_i \qquad k' = \sum N_i' \tag{13}$$

with additional conditions

$$\sum n_i = 2\lambda - k \qquad \sum n_i' = 2\lambda - k' \tag{14}$$

whereas the summation variables α_i, β_i, γ_i, δ_i and γ are limited by

$$\begin{aligned} &\alpha_i + \gamma_i = n_i && \beta_i + \delta_i = N_i \\ &\beta_i + \gamma_i = n_i' && \alpha_i + \delta_i = N_i' \\ &\sum \alpha_i = 2\lambda - k - \gamma && \sum \beta_i = 2\lambda - k' - \gamma \\ &\sum \gamma_i = \gamma && \sum \delta_i = k + k' + \gamma - 2\lambda \\ &\sup(0, 2\lambda - k - k') \leqslant \gamma \leqslant 2\lambda - \sup(k, k'). \end{aligned} \tag{15}$$

Numbers k and k' can be chosen to reach the irreducible representations $(\lambda_1, \lambda_2, 0)$ if

$$\begin{aligned} &k = k' = \sum N_i = \sum N_i' = \lambda_2 \\ &\sum n_i = \sum n_i' = \lambda_1 + \lambda_2 = 2\lambda - \lambda_2. \end{aligned} \tag{16}$$

When $(0, \lambda, 0)$ is the interesting representation, one must choose

$$k = k' = \lambda. \tag{17}$$

Then γ varies from 0 to λ and there are $(\lambda + 1)$ constants D_γ to be determined.

A sufficient set of conditions is

$$\left|\begin{matrix}\lambda & 0 & 0 & 0\\ S & \lambda-S & 0 & 0\end{matrix}\right\rangle = 0 \qquad \text{for } S=1,\ldots,\lambda. \tag{18}$$

Easy calculations lead to the conditions

$$\sum_{\lambda-S}^{\lambda}\binom{\gamma}{\lambda-S}D_\gamma = 0. \tag{19}$$

Finally,

$$D_\gamma = (-1)^\gamma\binom{\lambda}{\gamma} = \frac{(-1)^\gamma\lambda!}{(\lambda-\gamma)!\,\gamma!}, \tag{20}$$

as can be verified using the identity

$$\sum_{\gamma=0}^{S}\frac{(-1)^\gamma S!\,x^\gamma}{\gamma!\,(S-\gamma)!} = (1-x)^S. \tag{21}$$

4.3. The general case $(\lambda_1, \lambda_2, \lambda_3)$

This more complicated situation is happily not useful at present for physics. The way to handle it is via

$$(\lambda_1+\lambda_2, 0, 0)\otimes(\lambda_2, 0, 0)\otimes(0, 0, \lambda_3)$$

with a base

$$\left|\begin{matrix} n_i\\ N_i\\ m_i\end{matrix}\right\} = u^{n_1}d^{n_2}s^{n_3}c^{n_4}U^{N_1}D^{N_2}S^{N_3}C^{N_4}\bar{u}^{m_1}\bar{d}^{m_2}\bar{s}^{m_3}\bar{c}^{m_4} \tag{22}$$

$$\sum n_i = \lambda_1+\lambda_2 \qquad \sum N_i = \lambda_2 \qquad \sum m_i = \lambda_3.$$

In principle the two procedures described above are sufficient to define the projectors. It is, however, doubtful that a general form of their matrix elements could be produced. Each particular case is to be treated individually.

5. Some pay-off with direct product reductions

While reducing the direct product

$$(n, 0, m)\otimes(n', 0, m') = \bigoplus(\lambda_1, \lambda_2, \lambda_3), \tag{23}$$

it is sometimes useful to know how many times a given representation $(N, 0, M)$ is contained in the product without entirely reducing it.

For this purpose, compare the invariants built with some auxiliary vectors $\bar{A}$ and B and with the bases

$$x_i^{n_i}\bar{y}_i^{m_i}x_i'^{\,n_i'}\bar{y}_i'^{\,m_i'} \qquad X_i^{N_i}\bar{Y}_i^{M_i} \tag{24}$$

respectively of the product $\{n, 0, m\} \otimes \{n', 0, m'\}$ and of $\{N, 0, M\}$. This comparison leads to

$$(\bar{A}, X)^N (\bar{Y}, B)^M = \sum_{\alpha\ldots\rho} C_{\alpha\ldots\rho} (\bar{A}, B)^\alpha (\bar{y}, x)^\beta (\bar{y}', x')^\gamma (\bar{A}, x)^\delta \times (\bar{A}, x')^\epsilon (\bar{y}, B)^\lambda (\bar{y}, x')^\mu (\bar{y}', B)^\nu (\bar{y}', x)^\rho \tag{25}$$

with the conditions

$$\begin{aligned} &\beta + \delta + \rho = n \qquad && \beta + \lambda + \mu = m \\ &\gamma + \epsilon + \mu = n' && \gamma + \nu + \rho = m' \\ &\alpha + \delta + \epsilon = N && \alpha + \lambda + \nu = M. \end{aligned} \tag{26}$$

Projection on the irreducible $(n, 0, m)$, $(n', 0, m')$ and $(N, 0, M)$ brings in extra conditions

$$\alpha = \beta = \gamma = 0. \tag{27}$$

The number K of independent $(N, 0, M)$ representations in the product $(n, 0, m) \otimes (n', 0, m')$ is then given by the number of entire non-negative solutions of the system

$$\begin{aligned} &\delta + \rho = n \qquad && \epsilon + \mu = n' \qquad && \delta + \epsilon = N \\ &\lambda + \mu = m && \nu + \rho = m' && \lambda + \nu = M, \end{aligned} \tag{28}$$

from which comes

$$N - M = n - m + n' - m' \tag{29}$$

$$\begin{aligned} K &= n + 1 + \inf(0, m' - n, n' - N) + \inf(0, M - m', N - n) \\ &= m + 1 + \inf(0, n' - m, m' - M) + \inf(0, N - n', M - m) \\ &= n' + 1 + \inf(0, m - n', n - N) + \inf(0, M - m, N - n') \\ &= m' + 1 + \inf(0, N - n, M - m') + \inf(0, n, -m', m - M) \\ &= N + 1 + \inf(0, n' - N, m' - n) + \inf(0, m - n', n - N) \\ &= M + 1 + \inf(0, m' - M, n' - m) + \inf(0, n - m', m - M). \end{aligned} \tag{30}$$

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